2024

MATHEMATICS — HONOURS

Paper : CC-3

(Real Analysis)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote the set of all natural, rational and real numbers respectively.

Notations and symbols have their usual meanings.

1. Answer all the following multiple choice questions. For each question 1 mark is for choosing the correct option and 1 mark is for justification. (1+1)×10

(a) The derived set of the set $S=\left\{\frac{1}{n}: n\in\mathbb{N}\right\}$ is

(i) ϕ (ii) $\{0\}$

(iii) $[0, 1]\cap\mathbb{Q}$ (iv) $[0, 1]$.

(b) Let $S=(0, 3)$ and $T=\bigcap_{n=1}^{\infty}\left(1-\frac{1}{n}, 2+\frac{1}{n}\right)$. Then $S\cap(\mathbb{R}-T)$ is

(i) open but not closed (ii) closed but not open

(iii) both open and closed (iv) neither open nor closed.

(c) The set $\left\{\sqrt{2}+r\sqrt{3}: r\in\mathbb{Q}\right\}$ is

(i) uncountable (ii) enumerable

(iii) finite (iv) dense in $\mathbb{R}$.

(d) Let S be a non-empty subset of $\mathbb{R}$. Which of the following is true?

(i) If S is bounded, then S has a limit point.

(ii) If S is closed and $x\in S$, then x is a limit point of S.

(iii) If $x\notin S$, then x is an exterior point of S.

(iv) If S is open and $x\in S$, then x is a limit point of S.

(e) The sequence $\left\{\left(\frac{3}{4}\right)^n + \left(\frac{4}{5}\right)^n\right\}$

(i) diverges to ∞ (ii) converges to '2'

(iii) converges to $\frac{\text{'31'}}{20}$ (iv) converges to '0'.

(f) Let $u_n = \cos n\pi$. Then $\lim\limits_{n\to\infty} \sup u_n$ is equal to

(i) -2 (ii) 0

(iii) 1 (iv) -1.

(g) The sequence $\left\{\frac{1}{n} + n\right\}$

(i) converges to 0 (ii) converges to 1

(iii) diverges to ∞ (iv) is oscillatory.

(h) Which of the following sequences is not a Cauchy sequence?

(i) $\left\{\frac{(-1)^n}{n}\right\}$ (ii) $\left\{\frac{1}{2^n}\right\}$

(iii) $\{n^n\}$ (iv) $\left\{\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}\right\}$.

(i) If $\{a_n\}$ is a convergent sequence, then $\left\{\frac{\sum_{i=1}^{n} a_i}{n}\right\}$ is

(i) bounded but not necessarily convergent

(ii) convergent and converges to $\lim\limits_{n\to\infty} a_n$

(iii) divergent

(iv) oscillatory.

(j) The series $\left(\frac{1}{1.2}+\frac{1}{2.3}+...+\frac{1}{n(n+1)}+...\right)$ is

(i) a divergent series

(ii) convergent and sum is '0'

(iii) a convergent series and the sum is '1'

(iv) oscillatory.

Unit - 1

Answer ***any four*** questions.

2. (a) Prove and disprove : A countable set can never have uncountable number of limit points.

(b) Find all the isolated points of $\left\{\frac{n}{n+1}:n\in\mathbb{N}\right\}\cup(2,3]$. 3+2

3. Prove or disprove :

(a) If S, T are non-empty bounded subset of $\mathbb{R}$, then $\sup(S-T)=\sup S-\sup T$.

(b) The set $A=\{x+y : x\in\mathbb{Q},\ y\in\mathbb{R}\setminus\mathbb{Q}\}$ is countable. 3+2

4. (a) Prove or disprove : Every enumerable set has a limit point.

(b) Prove that every bounded open interval has a limit point. 2+3

5. (a) Prove or disprove : Every bounded infinite set has an interior point.

(b) Let a and b be two real numbers such that $a<b$. Show that there is a rational number q such that $a<q<b$. 2+3

6. Prove that $\mathbb{R}$ has only two subsets which are both open and closed. 5

7. (a) Prove that arbitrary union of open set is open.

(b) Show that the set S is an open set, where

$$S=\{x\in\mathbb{R} : x^2-5x+6>0\}.$$ 3+2

8. (a) Is the set $S=\left\{x\in\mathbb{R}:\sin\frac{1}{x}=0\right\}$ enumerable? Justify.

(b) Prove that the union of two enumerable sets is enumerable. 2+3

Please Turn Over

Unit - 2

Answer ***any four*** questions.

9. (a) Prove or disprove :

The sequence $\{x_n\}$ where $x_n = \frac{n}{2} - \left[\frac{n}{2}\right]$ is convergent. ([x] denotes the largest integer not exceeding x).

(b) Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

2+3

10. (a) Give examples of two non-convergent sequences $\{x_n\}$ and $\{y_n\}$ such that both the sequences $\{x_n y_n\}$ and $\left\{x_n/y_n\right\}$ are convergent.

(b) If for a sequence $\{x_n\}$ of real numbers $\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} x_{2n}$, then prove that $\{x_n\}$ is convergent.

2+3

11. Prove that every monotonically increasing sequence which is bounded above is convergent. What happens if the sequence is unbounded above? Justify.

3+2

12. State and prove Cauchy's first limit theorem. Show that $\lim_{n\to\infty} \dfrac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{n} = 0$. (1+3)+1

13. (a) Prove that product of two convergent sequences is convergent.

(b) $\{x_n\}$ and $\{y_n\}$ are two sequences such that $x_n \geq 1\ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty}(x_n - y_n) = 0$. Prove that $\lim_{n\to\infty} \frac{y_n}{x_n} = 1$.

3+2

14. Define Cauchy sequence. Show that the sequence $\{x_n\}$ is a Cauchy sequence, where

$$x_n = 1 - \frac{1}{2} + \frac{1}{3} + \ldots + (-1)^{n-1}\frac{1}{n}.$$

1+4

15. Define subsequence of a sequence of real numbers. Prove that a bounded sequence $\{x_n\}$ is convergent if and only if $\lim\sup x_n = \lim\inf x_n$.

1+4

Unit - 3

Answer ***any one*** question.

16. State and prove Leibnitz's test. Using it show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$ is convergent. (1+3)+1

17. (a) Prove or disprove :

If $\sum_{n} a_n$ is a convergent series of real numbers, then $\sum_{n} a_n^2$ is also convergent.

(b) Test convergence of the infinite series whose n-th term is $\frac{2^2.4^2.6^2...(2n)^2}{3^2.5^2.7^2...(2n-1)^2}$. 2+3

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